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TWO-SPATIAL DIMENSIONAL ELASTIC WAVE PROPAGATION BY THE THEORY OF CHARACTERISTICS

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Abstract—A method is presented by which the theory of characteristics is extended to include elastic waves in two-spatial dimensions. This method makes use of Hadamard's work on surfaces of discontinuity in the dependent variables and their derivatives.

A model is developed for a surface propagating in a linear elastic, isotropic, and homogeneous medium. Velocities and stresses of the material particles are assumed to be discontinuous in their first partial derivatives across the propagating surface. A transformation is used for this model by which the relevant discontinuity relations from Hadamard's theory are applied to the dynamical field equations. Application of this transformation results in a system of necessary dynamical conditions which then lead to the derivation of the characteristic equations for two-spatial dimensions.

1. INTRODUCTION

THE phenomenon of transient wave propagation in solids has created increased interest in recent years. Solutions have been sought to wave problems, particularly those which arise in fields of geophysics and space technology. These problems generally require the treatment of finite bodies subjected to concentrated impulsive loads on one or more of their boundaries. Waves generated under these conditions are inherently multi-dimensional.

For a wide range of practical cases multi-dimensional systems can be narrowed down to two-spatial dimensional waves. In the past, transform techniques were applied to the solution of two-spatial dimensional wave propagation problems in solids. Only few boundary value problems have been solved using these techniques and then only in the case of linear materials [1].

Recently, the theory of characteristics has been successfully applied to two-spatial dimensional dynamic elasticity problems [2] and [3]. The characteristic surfaces and characteristic equations were derived by the conventional method. The conventional directional derivative approach, used extensively in the past to solve hydrodynamic equations, is summarized and clearly presented in Reference [4].

The motivation for this study is to establish a method by which the characteristic surfaces and the characteristic equations will be derived by the employment of kinematical conditions which exist across a surface of discontinuity.

In his work on surfaces of discontinuity, Hadamard developed a theory which presents these surfaces as the loci of discontinuities in the dependent variables and their derivatives [5]. In view of this model, kinematical conditions were derived by Hadamard for the function and its derivatives in reference to space and time respectively, which are denoted as Hadamard's discontinuity relations. Since Hadamard's discontinuity relations were derived on kinematical basis only, they seem to be applicable to a wide variety of constitutive equations. This work was extended by Levi-Civita [6], T. Y. Thomas [7], and is also presented in References [8] and [9]. Therefore, it seems expedient to employ Hadamard's work in the effort to develop a suitable method for the solution of the two-spatial dimensional wave problems in solids.

The study which follows describes these efforts. Hadamard's theory is applied to a wave surface across the first partial derivatives of the velocities and stresses are discontinuous, while these dependent variables themselves are continuous. The discontinuity relations are reformulated in order to reduce the basic dynamical field equations to a system of dynamical conditions. The objective of this analysis is to present the dynamical conditions as a set of equations equal in number to the discontinuous first partial derivatives. From the dynamical conditions the differential equations governing the propagation of discontinuities on the wave surfaces are derived. These characteristic equations are derived with reference first to Cartesian and then to cylindrical coordinate systems, thus allowing their application to a wide range of problems.

The investigation described in this study is confined to two-spatial dimensional wave propagation in a linear elastic, isotropic, and homogeneous material.

2. BASIC DYNAMICAL FIELD EQUATIONS AND THE MONGE CONE

The basic dynamical field equations which govern the deformation in the medium are written below with respect to the Cartesian coordinate system. Total differentials of the dependent variables are then combined with the basic dynamical field equations to produce a characteristic surface.

The basic dynamical field equations for linear elastic, isotropic and homogeneous material where plane strain is prevalent are

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{zx}}{\partial z} = \rho\frac{\partial U}{\partial t} \tag{1}$$

$$\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z} = \rho\frac{\partial W}{\partial t} \tag{2}$$

$$\frac{\partial \sigma_{zx}}{\partial t} = \mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right) \tag{3}$$

$$\frac{\partial \sigma_{zz}}{\partial t} = \lambda\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}\right)+2\mu\frac{\partial W}{\partial z} \tag{4}$$

$$\frac{\partial \sigma_{xx}}{\partial t} = \lambda\left(\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z}\right)+2\mu\frac{\partial U}{\partial x} \tag{5}$$

where

ρ = density

U and W = particle velocities in the x and z directions respectively

σ_{xx}, σ_{zz} and σ_{xz} = the normal and shear stresses in the x, z plane

x and z = Cartesian coordinates

t = the time dimension

μ and λ = Lamé's constants.

In addition to the basic dynamical field equations (1) through (5), total differentials for the dependent variables are necessary to produce an integral surface as evident from the following: Let a function f stand for any of the dependent variables σ_{xx}, σ_{zz}, σ_{xz}, U, and W, and let f and its first partial derivatives $\partial f/\partial x$, $\partial f/\partial z$, and $\partial f/\partial t$ be continuous in certain regions in the medium which occupies the space x, z, t. It is known that at each point in the medium, where the basic dynamical field equations are defined, there exists a Monge cone [10]. The Monge cone is formed by varying the first partial derivatives $\partial f/\partial x$, $\partial f/\partial z$, and $\partial f/\partial t$, which are algebraically related by the basic dynamical field equations. For a surface $f = f(x, z, t)$ to be an integral surface of the basic dynamical field equations its tangent element should touch the Monge cone at each point in the field defined by the equations. The point $(x+\mathrm{d}x, z+\mathrm{d}z, t+\mathrm{d}t, f+\mathrm{d}f)$ lies in a tangent element to the Monge cone at the point x, z, t, f if $\mathrm{d}f = (\partial f/\partial x)\,\mathrm{d}x+(\partial f/\partial z)\,\mathrm{d}z+(\partial f/\partial t)\,\mathrm{d}t$ [10]. Consequently, the total differentials of the dependent variables must be introduced. Equations (1) through (5) and the total differentials of the dependent variables are combined to yield the following equations:

$$\rho\frac{\partial U}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\rho\frac{\partial U}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{zx}}{\partial z} = \rho\frac{\mathrm{d}U}{\mathrm{d}t} \tag{6}$$

$$\rho\frac{\partial W}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\rho\frac{\partial W}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\frac{\partial \sigma_{xz}}{\partial x}+\frac{\partial \sigma_{zz}}{\partial z} = \rho\frac{\mathrm{d}W}{\mathrm{d}t} \tag{7}$$

$$\frac{\partial \sigma_{zx}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial \sigma_{zx}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\mu\frac{\partial W}{\partial x}+\mu\frac{\partial U}{\partial z} = \frac{\mathrm{d}\sigma_{zx}}{\mathrm{d}t} \tag{8}$$

$$\frac{\partial \sigma_{zz}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial \sigma_{zz}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\lambda\frac{\partial U}{\partial x}+\lambda\frac{\partial W}{\partial z}+2\mu\frac{\partial W}{\partial z} = \frac{\mathrm{d}\sigma_{zz}}{\mathrm{d}t} \tag{9}$$

$$\frac{\partial \sigma_{xx}}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial \sigma_{xx}}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}+\lambda\frac{\partial U}{\partial x}+\lambda\frac{\partial W}{\partial z}+2\mu\frac{\partial U}{\partial x} = \frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t}. \tag{10}$$

This system of differential equations will be referred to as the dynamical field equations.

The dynamical field equations (6) through (10) are shown in the section "Characteristic Condition" to possess five real and distinct integral surfaces. Consequently, the dynamical field equations are of the hyperbolic type and their integral surfaces are known [10] as characteristic surfaces.

A characteristic surface starts to form when a point $P_0(x_0, z_0, t_0)$ in the medium, where the dynamical field equations are defined, is excited by one or more of the dependent variables. To show the formulation of an arbitrary characteristic surface by its tangent elements for an arbitrary point $P_0(x_0, z_0, t_0)$, the Monge cone with P_0 as its apex is considered (see Fig. 1). Since each of the tangent elements of the characteristic surface must touch the Monge cone at point P_0, these elements will form a pyramid around the Monge cone. The envelope of this pyramid is the characteristic surface. The generators of this envelope coincide with the pyramid along curves which are formed by the intersection of the tangent elements. These curves are known as the bicharacteristic curves. Since the bicharacteristic curves are formed by tangent elements with different orientations, discontinuities may occur in the first partial derivatives across the bicharacteristic curves. Therefore, the first partial derivatives on the characteristic surface are indeterminate. Accordingly, the characteristic surfaces are surfaces of discontinuity in the first partial derivatives of the dependent variables. When point P_0 is excited by given initial conditions

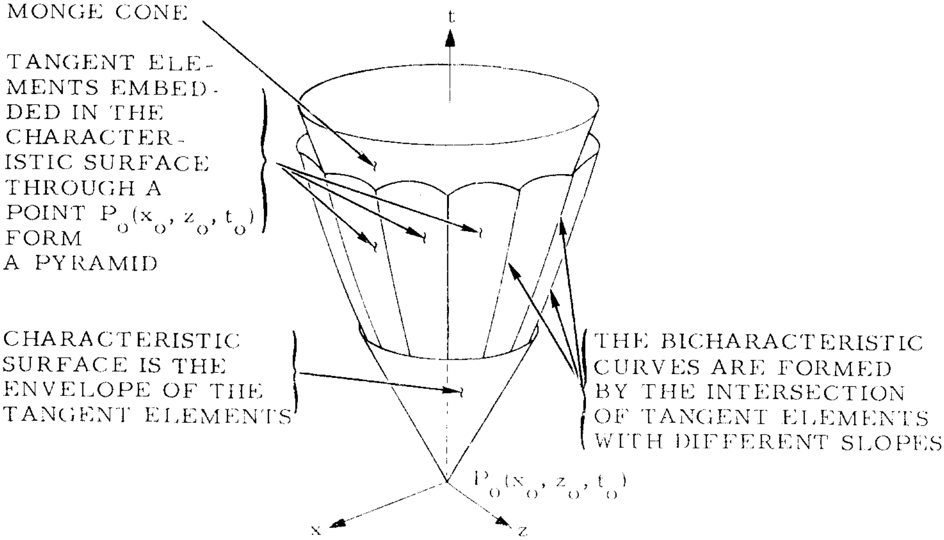

FIG. 1. Characteristic surfaces and bicharacteristic curves.

which are functions of time, the surface discontinuity is then propagated in the space x, z, t as shown in Fig. 1, and so are the indeterminate first partial derivatives along the bicharacteristic curves. The solution surface is then found by the propagation of these indeterminate first partial derivatives from the initial conditions along the bicharacteristic curves.

The objectives now are first to depict the surface of discontinuity as a wave surface and then to establish the dynamical conditions for the indeterminate first partial derivatives. The dynamical conditions are to be established from the dynamical field equations (6) through (10). From the dynamical conditions a characteristic condition is to be found and the latter should yield the discontinuity surfaces and their bicharacteristic curves along which the indeterminate first partial derivatives are propagating. Next, the characteristic equations which relate the dependent variables along the bicharacteristic curves are to be derived. Once these objectives have been reached, the propagation of the solution surface can be found when a point is excited by given initial conditions. The following sections are devoted to these objectives.

3. WAVES AS SURFACES OF DISCONTINUITY

Analytical tools are now being sought to treat waves as surfaces of discontinuity. The discussion in the end of the previous section described the geometrical meaning and the formulation of surfaces of discontinuity from the dynamical field equations (6) through (10). With this notion, a model is defined here where waves are depicted as surfaces of discontinuity. Once this model is defined, the theory on surfaces of discontinuity based on Hadamard [5] is introduced to present the discontinuity relations which hold across the wave surface.

The model considered is a surface $\Phi(x, z, t) = 0$ in the medium. Two basic conditions are imposed on this surface:

(a) It is required of the surface to present the locus of possible discontinuities in the first partial derivatives of the dependent variables appearing in the dynamical field equations. The dependent variables themselves are assumed to be continuous everywhere, but differentiable only in regions which are in the rear and in the front of the discontinuity surface.

(b) It is required of this surface of discontinuity to present a wave surface in the sense that while moving in the medium it affects different material particles as it moves.

From these basic conditions and from the dynamical field equations (6) through (10) it follows that the quantities $\partial U/\partial x$, $\partial U/\partial z$, $\partial W/\partial x$, $\partial W/\partial z$, and $\sigma_{ij,i}$ across the wave surface Φ are assumed to be discontinuous, but ρ, σ_{ij}, U and W are assumed to be continuous.

Rather than a direct treatment of equations (6) through (10), which implies a direct dynamical analysis, it is first desired to confine the analysis to the kinematical conditions for the indeterminate first partial derivatives across the wave surface.

In his work on surfaces of discontinuity Hadamard developed Kinematical conditions, which are also known as discontinuity relations, for the dependent variables and their derivatives across the surface. These analytical tools, based on Hadamard, are adapted here to formulate the proposed wave surface model. Corresponding to the basic conditions imposed on the surface of discontinuity, the relevant Hadamard's discontinuity relations [5] are written as follows:

$$[\sigma_{ij,j}] = \delta_{ij}n_j, \quad \left[\frac{\partial U}{\partial x}\right] = \delta_x n_x, \quad \left[\frac{\partial U}{\partial z}\right] = \delta_x n_z, \quad \left[\frac{\partial W}{\partial x}\right] = \delta_z n_x$$

$$\left[\frac{\partial W}{\partial z}\right] = \delta_z n_z \quad \text{and} \quad [\sigma_{ij}] = [U] = [W] = [\rho] = 0 \tag{11}$$

where, if f stands for any of the dependent variables (see Fig. 2), then $[f_{,i}]$ denotes the discontinuity in the first partial derivative across the wave surface $\Phi(x, z, t) = 0$. The quantity

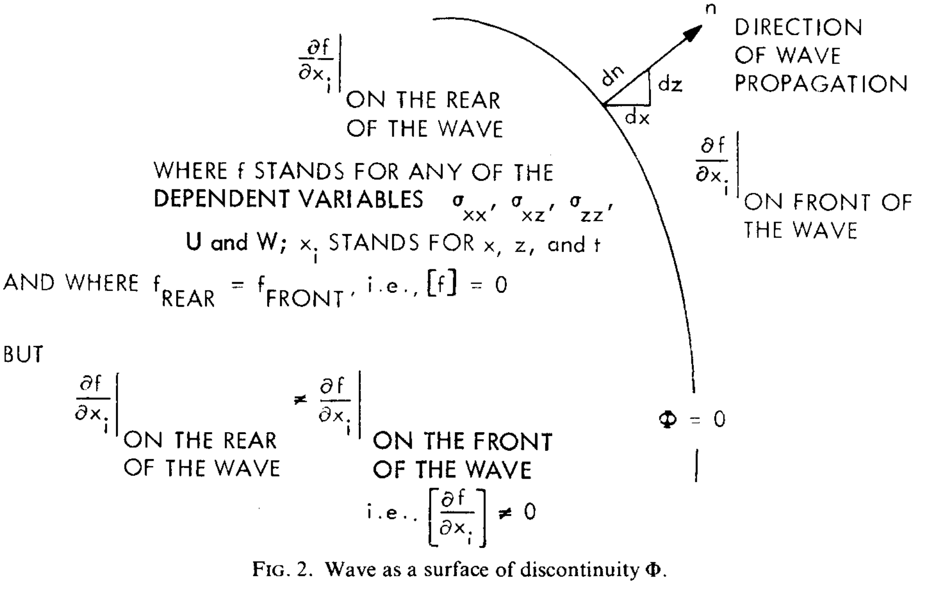

FIG. 2. Wave as a surface of discontinuity Φ.

$[f_{,i}]$ is defined [7] to be the value of the first partial derivative on the rear of the wave surface minus its value on the front of the wave surface, i.e.

$$[f_{,i}] = f_{,i}\Big|_{\substack{\text{The value of } f_{,i} \\ \text{on the rear of } \Phi}} - f_{,i}\Big|_{\substack{\text{The value of } f_{,i} \\ \text{on the front of } \Phi}} \tag{12}$$

δ_{ij}, δ_x and δ_z are arbitrary functions related to σ_{ij}, U and W respectively. n_x and n_z are the Cartesian components of the unit normal vector $\mathbf{n}$ to the wave surface Φ, such that

$$n_x^2 + n_z^2 = 1 \quad \text{let} \quad n_x = \cos(n, x) \quad \text{and} \quad n_z = \sin(n, x). \tag{13}$$

To summarize, a wave surface was considered across which discontinuities may occur in the first partial derivatives of the dependent variables. For this wave surface model, discontinuity relations, based on the work of Hadamard, are provided. It is now required to incorporate these discontinuity relations into the dynamical field equations so that the dynamical conditions for the indeterminate first partial derivatives can be obtained. This is done in the next section where a transformation is used for this purpose.

4. DYNAMICAL CONDITIONS

Dynamical conditions for the indeterminate first partial derivatives are derived in this section. To present these conditions as equations which relate the indeterminate first partial derivatives in a compatible manner, a transformation is used by which the discontinuity relations are applied to the dynamical field equations.

The discontinuity relations expressed in equations (11) are now transformed to a form where first partial derivatives appear with respect to the normal of the wave surface. Consequently, equations (11) are rewritten in the following form [8]:

$$[f_{,i}] = \left[\frac{\partial f}{\partial n}\right] n_i \tag{14}$$

or, by definition (12) equation (14) can further be written as follows:

$$f_{,i}\Big|_{\substack{\text{The value of } f_{,i} \text{ on the} \\ \text{rear of the wave surface}}} = \left[\frac{\partial f}{\partial n}\right] n_i + f_{,i}\Big|_{\substack{\text{The value of } f_{,i} \text{ on the} \\ \text{front of the wave surface}}} \tag{15}$$

where the value of $f_{,i}$ in front of the wave is finite and known from prescribed conditions. This is not the case, however, for the $f_{,i}$ on the rear of the wave surface. The first partial derivatives on the rear of the wave surface are created as the wave surface is being formed by the intersection of its tangent elements. Therefore, the $f_{,i}$ on the rear of the wave surface is considered here to be indeterminate. Equation (15) is rewritten as follows:

$$f_{,i}\Big|_{\substack{\text{Indeterminate} \\ \text{value of } f_{,i}}} = \left[\frac{\partial f}{\partial n}\right] n_i + f_{,i}. \tag{16}$$

The analysis done so far has been confined to kinematical conditions for the indeterminate first partial derivatives of the dependent variables on the wave surface. These kinematical conditions, expressed in equations (16) are now applied to dynamical field equations. In accordance with Section 2, the first partial derivatives in the dynamical field equations correspond to the left-hand side terms of equations (16) across a wave surface. Therefore, these derivatives are replaced by the right-hand side terms of equations (16) and equations (6) through (10) yield the following relations:

$$\begin{aligned}
\rho\left[\frac{\partial U}{\partial n}\right]\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\left[\frac{\partial \sigma_{xx}}{\partial n}\right]n_x+\left[\frac{\partial \sigma_{zx}}{\partial n}\right]n_z &= R_U\\
\rho\left[\frac{\partial W}{\partial n}\right]\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\left[\frac{\partial \sigma_{zx}}{\partial n}\right]n_x+\left[\frac{\partial \sigma_{zz}}{\partial n}\right]n_z &= R_W\\
\left[\frac{\partial \sigma_{xz}}{\partial n}\right]\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\mu\left[\frac{\partial W}{\partial n}\right]n_x+\mu\left[\frac{\partial U}{\partial n}\right]n_z &= R_{xz}\\
\left[\frac{\partial \sigma_{zz}}{\partial n}\right]\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\lambda\left[\frac{\partial U}{\partial n}\right]n_x+(\lambda+2\mu)\left[\frac{\partial W}{\partial n}\right]n_z &= R_{zz}\\
\left[\frac{\partial \sigma_{xx}}{\partial n}\right]\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)+\lambda\left[\frac{\partial W}{\partial n}\right]n_z+(\lambda+2\mu)\left[\frac{\partial U}{\partial n}\right]n_x &= R_{xx}
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
R_U &= \rho\frac{\mathrm{d}U}{\mathrm{d}t}-\rho\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(\frac{\partial U}{\partial x}n_x+\frac{\partial U}{\partial z}n_z\right)-\frac{\partial \sigma_{xx}}{\partial x}-\frac{\partial \sigma_{xz}}{\partial z}\\
R_W &= \rho\frac{\mathrm{d}W}{\mathrm{d}t}-\rho\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(\frac{\partial W}{\partial x}n_x+\frac{\partial W}{\partial z}n_z\right)-\frac{\partial \sigma_{zz}}{\partial z}-\frac{\partial \sigma_{xz}}{\partial x}\\
R_{xz} &= \frac{\mathrm{d}\sigma_{xz}}{\mathrm{d}t}-\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(\frac{\partial \sigma_{xz}}{\partial x}n_x+\frac{\partial \sigma_{xz}}{\partial z}n_z\right)-\mu\left(\frac{\partial W}{\partial x}+\frac{\partial U}{\partial z}\right)\\
R_{zz} &= \frac{\mathrm{d}\sigma_{zz}}{\mathrm{d}t}-\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(\frac{\partial \sigma_{zz}}{\partial x}n_x+\frac{\partial \sigma_{zz}}{\partial z}n_z\right)-(\lambda+2\mu)\frac{\partial W}{\partial z}-\lambda\frac{\partial U}{\partial x}\\
R_{xx} &= \frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}t}-\left(n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t}\right)\left(\frac{\partial \sigma_{xx}}{\partial x}n_x+\frac{\partial \sigma_{xx}}{\partial z}n_z\right)-(\lambda+2\mu)\frac{\partial U}{\partial x}-\lambda\frac{\partial W}{\partial z}.
\end{aligned} \tag{18}$$

Let

$$G = n_x\frac{\mathrm{d}x}{\mathrm{d}t}+n_z\frac{\mathrm{d}z}{\mathrm{d}t} \tag{19}$$

which is recognized as the common term in the system of equations (17) and (18).

Rewriting equations (17) in matrix form gives

$$\begin{bmatrix}
\rho G & 0 & n_x & 0 & n_z\\
0 & \rho G & 0 & n_z & n_x\\
\mu n_z & \mu n_x & 0 & 0 & G\\
\lambda n_x & (\lambda+2\mu)n_z & 0 & G & 0\\
(\lambda+2\mu)n_x & \lambda n_z & G & 0 & 0
\end{bmatrix}
\begin{Bmatrix}
\left[\dfrac{\partial U}{\partial n}\right]\\
\left[\dfrac{\partial W}{\partial n}\right]\\
\left[\dfrac{\partial \sigma_{xx}}{\partial n}\right]\\
\left[\dfrac{\partial \sigma_{zz}}{\partial n}\right]\\
\left[\dfrac{\partial \sigma_{xz}}{\partial n}\right]
\end{Bmatrix}
=
\begin{Bmatrix}
R_U\\ R_W\\ R_{xz}\\ R_{zz}\\ R_{xx}
\end{Bmatrix}. \tag{20}$$

Hence, equation (20) presents a system of five equations which relate five first partial derivatives

$$\left[\frac{\partial U}{\partial n}\right], \quad \left[\frac{\partial W}{\partial n}\right], \quad \left[\frac{\partial \sigma_{xx}}{\partial n}\right], \quad \left[\frac{\partial \sigma_{zz}}{\partial n}\right] \quad \text{and} \quad \left[\frac{\partial \sigma_{zx}}{\partial n}\right]$$

of the five dependent variables U, W, σ_{xx}, σ_{zz} and σ_{xz}. The system of differential equations (20) is then denoted as the dynamical conditions, since it is from these relations that the wave surfaces and the characteristic equations will be derived.

In order to derive the wave surfaces and then the characteristic equations which hold on these surfaces from the dynamical conditions, the following conventional approach is attempted. To solve any of the unknowns, say $[\partial U/\partial n]$, in equation (20), a ratio is needed where its denominator is the determinant of the coefficients and its numerator is the determinant where the determinant's first column is replaced by the right-hand column of equation (20). However, since it is required of $[\partial U/\partial n]$ to be discontinuous, the denominator is made to vanish. Furthermore, since it was established that this discontinuity takes place across the wave surface, it can be concluded that by equating the denominator to zero the wave surface can be determined. Therefore, the determinant of the coefficients when equated to zero is denoted as the characteristic condition. On the other hand, it is also required of the discontinuities to be related in order to define the tangent elements (refer to the section "Basic Dynamical Field Equations and the Monge Cone") of the wave surface. This is established when the numerator is also made to vanish. The numerator, which is equated to zero, and the equations for the characteristic surfaces will yield the characteristic equations.

In view of the above description of the dynamical conditions, the following steps are taken to determine

(a) the wave surface (or surfaces) from the characteristic condition and the generators of the surfaces, which are the bicharacteristic curves (this is done in the next section);
(b) the characteristic equation (or equations) which hold along the bicharacteristic curves (this is done in the section "Cartesian Characteristic Equations").

5. CHARACTERISTIC CONDITION AND BICHARACTERISTIC CURVES

Once the dynamical conditions for the indeterminate first partial derivatives have been established, it is possible to proceed to determine the wave surfaces from the characteristic condition. The bicharacteristic curves are then derived from these characteristic surfaces.

The determinant of the coefficients of the matrix (20) is equated to zero in order to establish the characteristic condition, and is expanded to yield five distinct and real roots as follows:

$$G_1 = \sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)} \tag{21}$$

$$G_2 = -\sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)} \tag{22}$$

$$G_3 = \sqrt{\left(\frac{\mu}{\rho}\right)} \tag{23}$$

$$G_4 = -\sqrt{\left(\frac{\mu}{\rho}\right)} \tag{24}$$

$$G_5 = 0. \tag{25}$$

In order to interpret the meaning of these roots, a wave velocity **g** is defined as follows:

Let Φ be the wave surface and let a material particle P be on this surface at time t. At time $t + dt$ the normal **n** at P to $\Phi(t)$ will intersect the same wave surface at a different material particle P' (see Fig. 3). Hence, the velocity **g** of the wave surface Φ at P at the instant t is defined to be [5]

$$\mathbf{g} = \frac{\mathbf{PP'}}{dt}$$

or the wave surface speed is

$$g = \frac{dn}{dt} \tag{26}$$

where it is apparent from Fig. 3 and from equation (13) that dn may be presented by vector addition as

$$dn = n_x\, dx + n_z\, dz \tag{27}$$

or

$$g = n_x \frac{dx}{dt} + n_z \frac{dz}{dt}.$$

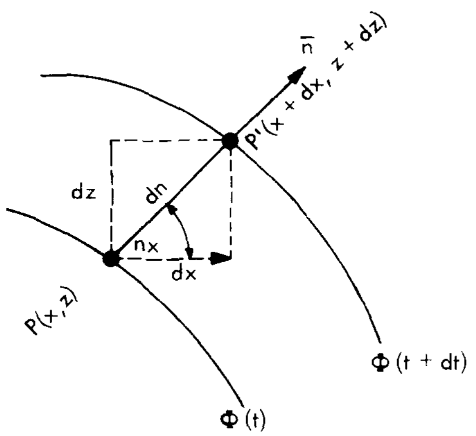

FIG. 3. The wave surface Φ.

This expression is identical to the relation expressed in equation (19). Therefore, it is clear that G is identical to g, which is the propagation speed of the wave surface. This is also consistent with the dimensions of G [equations (21) through (24)]. Furthermore, from Fig. 3 it is also apparent that

$$(dn)^2 = (dx)^2 + (dz)^2$$

or in view of the definition (26) that

$$(dx)^2 + (dz)^2 = G^2(dt)^2. \tag{28}$$

Equation (28) defines the Monge cone [10] (see Fig. 1). Each wave surface through the apex of this cone touches this cone along the bicharacteristic curves. The bicharacteristic curves are easily found by examining equation (28) in Fig. 4.

In view of equation (28) and Fig. 4 it is apparent that the bicharacteristic curves may be presented as follows:

$$\begin{aligned} \mathrm{d}x &= G\cos\theta\,\mathrm{d}t \\ \mathrm{d}z &= G\sin\theta\,\mathrm{d}t \end{aligned} \tag{29}$$

or

$$\begin{aligned} \mathrm{d}x &= Gn_x\,\mathrm{d}t \\ \mathrm{d}z &= Gn_z\,\mathrm{d}t. \end{aligned}$$

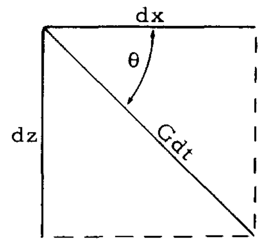

FIG. 4. Representation of equation (28).

The next task is to derive the characteristic equations, which relate the dependent variables along these bicharacteristic curves. Before this is done, however, some conclusions should be made as to the dynamical nature of the obtained wave speeds as expressed in equations (21) through (25).

First, it is intended to examine the nature of equation (25), i.e. $G_5 = 0$. For this purpose the equations of motion (6) and (7) are rewritten for the wave surface, in which case they include the discontinuity notations and the relations $\mathrm{d}x/\mathrm{d}t = Gn_x$ and $\mathrm{d}z/\mathrm{d}t = Gn_z$ (see Fig. 3) as follows:

$$\left[\frac{\partial\sigma_{xx}}{\partial x}\right]+\left[\frac{\partial\sigma_{xz}}{\partial z}\right] = -\rho G\left[\frac{\partial U}{\partial x}\right]n_x-\rho G\left[\frac{\partial U}{\partial z}\right]n_z+\rho\left[\frac{\mathrm{d}U}{\mathrm{d}t}\right]$$

$$\left[\frac{\partial\sigma_{xz}}{\partial x}\right]+\left[\frac{\partial\sigma_{zz}}{\partial z}\right] = -\rho G\left[\frac{\partial W}{\partial x}\right]n_x-\rho G\left[\frac{\partial W}{\partial z}\right]n_z+\rho\left[\frac{\mathrm{d}W}{\mathrm{d}t}\right].$$

Obviously, since U and W are assumed to be continuous across the wave surface then

$$\left[\frac{\mathrm{d}U}{\mathrm{d}t}\right] = \left[\frac{\mathrm{d}W}{\mathrm{d}t}\right] = 0.$$

When zero is substituted for G in the above equations, it can be seen that this root represents a front where static deformation takes place.

In order to interpret the significance of the roots $G^2 = (\lambda+2\mu)/\rho$ and $G^2 = \mu/\rho$ the discontinuity relations (11) are applied to equations (6) through (10) which are combined to yield the following two relations:

$$\begin{aligned} n_x(\lambda+\mu)(n_x\delta_x+n_z\delta_z) &= (\rho G^2-\mu)\delta_x \\ n_z(\lambda+\mu)(n_x\delta_x+n_z\delta_z) &= (\rho G^2-\mu)\delta_z. \end{aligned} \tag{30}$$

When $\lambda+2\mu/\rho$ is substituted for G^2 in equations (30) the following features are obtained:

$$\begin{aligned} n_x(n_x\delta_x+n_z\delta_z) &= \delta_x \\ n_z(n_z\delta_z+n_x\delta_x) &= \delta_z. \end{aligned} \tag{31}$$

These relations imply that the vector $\boldsymbol{\delta}$ is parallel to vector $\mathbf{n}$ or that the material particle at the wave surface moves parallel to the propagating wave. Also, the linearized vorticity at the wave surface is $\frac{1}{2}([\partial U/\partial z]-[\partial W/\partial x])$. In view of relations (11)

$$\left[\frac{\partial U}{\partial z}\right]-\left[\frac{\partial W}{\partial x}\right] = \delta_x n_z-\delta_z n_x$$

but as a result of the relations (31) the vorticity at the wave surface vanishes. A wave characterized by these features is known as the longitudinal wave, which propagates with speeds expressed in equations (21) and (22). G_1 and G_2 are now replaced by the notations G_L^+ and G_L^- respectively where the subscript L refers to longitudinal waves. This observation as well as the following one are restatements of T. Y. Thomas [11].

The substitution of μ/ρ into equation (24) yields the following relation:

$$n_x\delta_x+n_z\delta_z = 0 \tag{32}$$

which implies

$$\frac{\partial U}{\partial x}+\frac{\partial W}{\partial z} = 0.$$

Thus, the dilatation at the wave vanishes. Also, in view of relations (30) the term $n_x\delta_x+n_z\delta_z$ represents a scalar which may be written as a dot product. In conjunction with relation (32) this term becomes $\mathbf{n}\,.\,\boldsymbol{\delta} = 0$. A wave consisting of these features is known as the shear wave which propagates with the speeds expressed in equations (23) and (24). The speeds G_3 and G_4 are now replaced by the notations G_S^+ and G_S^- respectively where the subscript S refers to shear waves.

Thus far, the wave surfaces, their speeds of propagation, and their bicharacteristic curves have been established. In the next section the characteristic equations which hold along the bicharacteristic curves and on the wave surfaces will be derived.

6. CARTESIAN CHARACTERISTIC EQUATIONS

In this section the characteristic equations are derived from the dynamical conditions. As was described in the section "Dynamical Conditions," the procedure to obtain the characteristic equations is as follows: Each column in the coefficient determinant of equation (20) is replaced consecutively by the column on the right-hand side. The determinant is then made to vanish for each such replacement. The expansion of the determinant yields the desired relation.

The characteristic surfaces as expressed in relations (21) through (25) are substituted for G. The result is five distinct characteristic equations which relate the five dependent variables σ_{xx}, σ_{zz}, σ_{xz}, U and W along the bicharacteristic curves expressed in equation (29). These characteristic equations are:

$$n_z^2R_{zz}+n_x^2R_{xx}+2n_xn_zR_{zx}-G_L^+(n_zR_W+n_xR_U) = 0 \tag{33}$$

along

$$dx = G_L^+ n_x \, dt$$
$$dz = G_L^+ n_z \, dt$$

where

$$G_L^+ = +\sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)}$$

$$n_z^2 R_{zz} + n_x^2 R_{xx} + 2n_x n_z R_{xz} - G_L^- (n_z R_W + n_x R_U) = 0 \tag{34}$$

along

$$dx = G_L^- n_x \, dt$$
$$dz = G_L^- n_z \, dt$$

where

$$G_L^- = -\sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)}$$

$$n_x n_z (R_{zz} - R_{xx}) + (1-2n_z^2) R_{xz} - G_s^+ (n_x R_W - n_z R_U) = 0 \tag{35}$$

along

$$dx = G_S^+ n_x \, dt$$
$$dz = G_S^+ n_z \, dt$$

where

$$G_S^+ = +\sqrt{\left(\frac{\mu}{\rho}\right)}$$

$$n_x n_z (R_{zz} - R_{xx}) + (1-2n_z^2) R_{xz} - G_s^- (n_x R_W - n_z R_U) = 0 \tag{36}$$

along

$$dx = G_S^- n_x \, dt$$
$$dz = G_S^- n_z \, dt$$

where

$$G_S^- = -\sqrt{\left(\frac{\mu}{\rho}\right)}$$

$$[\lambda n_x^2 - (\lambda+2\mu) n_z^2] R_{xx} + [\lambda n_z^2 - (\lambda+2\mu) n_x^2] R_{zz} + 4 n_x n_z (\lambda+\mu) R_{xz} = 0 \tag{37}$$

along

$$dx = 0, \qquad dz = 0$$

where the R's are expressed in equations (18) when $n_x \, dx/dt + n_x \, dz/dt$ is replaced by the relevant G.

The characteristic equations (33) through (37) derived here are in complete agreement with the equations derived by Clifton [2]. Also, it can be shown that equations (33) through (37) contain upon reduction known one-spatial dimensional cases which require Cartesian characteristic equations. The next section is devoted to the derivation of the characteristic equations in the cylindrical coordinate system.

7. CYLINDRICAL CHARACTERISTIC EQUATIONS

This section is presented in order to accommodate many practical problems where the cylindrical coordinate system is advantageous. A system of five differential equations which relate the five unknown functions σ_{rr}, $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, U_r and U_θ along the bicharacteristic curves is to be derived here. The principles of the theory and the derivations are analogous to the analysis used to derive equations (33) through (37), and reference should be made to the previous sections.

The basic dynamical field equations for linear elastic material where plane strain is prevalent are

$$\frac{\partial \sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} = \rho\frac{\partial U_r}{\partial t}$$

$$\frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta}+\frac{\partial \sigma_{r\theta}}{\partial r}+\frac{2\sigma_{r\theta}}{r} = \rho\frac{\partial U_\theta}{\partial t}$$

$$\frac{\partial \sigma_{rr}}{\partial t} = \lambda\left(\frac{\partial U_r}{\partial r}+\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\theta}{\partial \theta}\right)+2\mu\frac{\partial U_r}{\partial r}$$

$$\frac{\partial \sigma_{\theta\theta}}{\partial t} = \lambda\left(\frac{\partial U_r}{\partial r}+\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\theta}{\partial \theta}\right)+2\mu\left(\frac{U_r}{r}+\frac{1}{r}\frac{\partial U_\theta}{\partial \theta}\right)$$

$$\frac{\partial \sigma_{r\theta}}{\partial t} = \mu\left(\frac{\partial U_\theta}{\partial r}-\frac{U_\theta}{r}+\frac{1}{r}\frac{\partial U_r}{\partial \theta}\right)$$

where

U_r and U_θ = radial and tangential displacements respectively
σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ = stress components in the r, θ plane
r and θ = plane cylindrical coordinates
t = the time dimension.

The kinematical conditions expressed in equations (16) are used here where the components of the unit vector normal to the wave surface are n_r and n_θ such that $n_r^2+n_\theta^2/r^2 = 1$.

By the same steps as for the Cartesian coordinate system, five characteristics are obtained which are identical to equations (21) through (25). The bicharacteristic curves may be found simply by transferring equation (28) into cylindrical coordinates, namely, $(\mathrm{d}r^2)+r^2(\mathrm{d}\theta)^2 = G^2(\mathrm{d}t)^2$. The bicharacteristic curves are $\mathrm{d}r = n_r G\,\mathrm{d}t$ and $\mathrm{d}\theta = n_\theta/r^2\, G\,\mathrm{d}t$. Now, following similar steps (as described in the section "Cartesian Characteristic Equations") one obtains the desired five characteristic equations along the bicharacteristics, which are the following equations.

$$
\begin{aligned}
&n_r^2 R_{rr} + \frac{n_\theta^2}{r^2} R_{\theta\theta} + 2\frac{n_r n_\theta}{r} R_{r\theta} - G_L n_r R_{U_r} - G_L \frac{n_\theta}{r} R_{U_\theta} \\
&\quad = \frac{(n_r\,\mathrm{d}r + n_\theta\,\mathrm{d}\theta)}{r\,\mathrm{d}t} \cdot \left[(\sigma_{\theta\theta} - \sigma_{rr}) n_r - 2\sigma_{r\theta}\frac{n_\theta}{r}\right] + \frac{1}{r}\left[\left(\lambda + 2\mu\frac{n_\theta^2}{r^2}\right) U_r - \frac{2n_r n_\theta}{r}\mu U_\theta\right]
\end{aligned}
\tag{38}
$$

along

$$\frac{\mathrm{d}r}{\mathrm{d}t} = G_L n_r$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = G_L \frac{n_\theta}{r^2}$$

$$
\begin{aligned}
&n_r^2 R_{rr} + \frac{n_\theta^2}{r^2} R_{\theta\theta} + 2\frac{n_r n_\theta}{r} R_{r\theta} + G_L n_r R_{U_r} + G_L \frac{n_\theta}{r} R_{U_\theta} \\
&\quad = \frac{(n_r\,\mathrm{d}r + n_\theta\,\mathrm{d}\theta)}{r\,\mathrm{d}t} \left[(\sigma_{\theta\theta} - \sigma_{rr}) n_r - 2\sigma_{r\theta}\frac{n_\theta}{r}\right] + \frac{1}{r}\left[\left(\lambda + 2\mu\frac{n_\theta^2}{r^2}\right) U_r - \frac{2n_r n_\theta}{r}\mu U_\theta\right]
\end{aligned}
\tag{39}
$$

along

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -G_L n_r$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -G_L \frac{n_\theta}{r^2}$$

$$
\begin{aligned}
&\frac{n_\theta n_r}{r} R_{rr} - \frac{n_\theta n_r}{r} R_{\theta\theta} - \left(n_r^2 - \frac{n_\theta^2}{r^2}\right) R_{r\theta} - G_S \frac{n_\theta}{r} R_{U_r} + G_S n_r R_{U_\theta} \\
&\quad = \frac{(n_{r\,\mathrm{d}r} + n_{\theta\,\mathrm{d}\theta})}{r\,\mathrm{d}t} \left[(\sigma_{\theta\theta} - \sigma_{rr})\frac{n_\theta}{r} + 2\sigma_{r\theta} n_r - 2U_r \rho G_S \frac{n_r n_\theta}{r} + U_\theta \rho G_S \left(n_r^2 - \frac{n_\theta^2}{r^2}\right)\right]
\end{aligned}
\tag{40}
$$

along

$$\frac{\mathrm{d}r}{\mathrm{d}t} = G_S n_r$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = G_S \frac{n_\theta}{r^2}$$

$$
\begin{aligned}
&\frac{n_\theta n_r}{r} R_{rr} - \frac{n_\theta n_r}{r} R_{\theta\theta} - \left(n_r^2 - \frac{n_\theta^2}{r^2}\right) R_{r\theta} + G_S \frac{n_\theta}{r} R_{U_r} - G_S n_r R_{U_\theta} \\
&\quad = \frac{(n_{r\,\mathrm{d}r} + n_{\theta\,\mathrm{d}\theta})}{r\,\mathrm{d}t} \left[(\sigma_{\theta\theta} - \sigma_{rr})\frac{n_\theta}{r} + 2\sigma_{r\theta} n_r + 2U_r \rho G_S \frac{n_r n_\theta}{r} - U_\theta \rho G_S \left(n_r^2 - \frac{n_\theta^2}{r^2}\right)\right]
\end{aligned}
\tag{41}
$$

along

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -G_S n_r$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -G_S \frac{n_\theta}{r^2}$$

$$\left[\lambda n_r^2-\frac{n_\theta^2}{r^2}(\lambda+2\mu)\right]R_{rr}+\left[\lambda\frac{n_\theta^2}{r^2}-n_r^2(\lambda+2\mu)\right]R_{\theta\theta}+4\frac{n_r n_\theta}{r}(\lambda+\mu)R_{r\theta}$$
$$=\frac{1}{r}\left\{U_r\left[\frac{n_\theta^2}{r^2}\lambda(\lambda+2\mu)-4\mu n_r^2(\lambda+\mu)\right]+4\frac{U_\theta}{r}n_r n_\theta\mu(\lambda+\mu)\right\} \tag{42}$$

along

$$\mathrm{d}r = 0$$
$$\mathrm{d}\theta = 0$$

where

$$G_L = \sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)}$$
$$G_S = \sqrt{\left(\frac{\mu}{\rho}\right)} \tag{43}$$

and where

$$R_{U_r} = \rho\frac{\mathrm{d}U_r}{\mathrm{d}t}-\rho G\left(\frac{\partial U_r}{\partial r}n_r+\frac{1}{r^2}\frac{\partial U_r}{\partial\theta}n_\theta\right)-\frac{\partial\sigma_{rr}}{\partial r}-\frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta}$$

$$R_{U_\theta} = \rho\frac{\mathrm{d}U_\theta}{\mathrm{d}t}-\rho G\left(\frac{\partial U_\theta}{\partial r}n_r+\frac{1}{r^2}\frac{\partial U_\theta}{\partial\theta}n_\theta\right)-\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}-\frac{\partial\sigma_{r\theta}}{\partial r}$$

$$R_{rr} = \frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}t}-G\left(\frac{\partial\sigma_{rr}}{\partial r}n_r+\frac{1}{r^2}\frac{\partial\sigma_{rr}}{\partial\theta}n_\theta\right)-(\lambda+2\mu)\frac{\partial U_r}{\partial r}-\frac{\lambda}{r}\frac{\partial U_\theta}{\partial\theta}$$

$$R_{\theta\theta} = \frac{\mathrm{d}\sigma_{\theta\theta}}{\mathrm{d}t}-G\left(\frac{\partial\sigma_{\theta\theta}}{\partial r}n_r+\frac{1}{r^2}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}n_\theta\right)-(\lambda+2\mu)\frac{1}{r}\frac{\partial U_\theta}{\partial\theta}-\lambda\frac{\partial U_r}{\partial r}$$

$$R_{r\theta} = \frac{\mathrm{d}\sigma_{r\theta}}{\mathrm{d}t}-G\left(\frac{\partial\sigma_{r\theta}}{\partial r}n_r+\frac{1}{r^2}\frac{\partial\sigma_{r\theta}}{\partial\theta}n_\theta\right)-\mu\left(\frac{\partial U_\theta}{\partial r}+\frac{1}{r}\frac{\partial U_r}{\partial\theta}\right).$$

A known one-spatial dimensional problem is considered in order to measure the validity of these equations. The problem consists of a circular cylindrical cavity in an infinite elastic solid subjected to a uniform radial load applied to the cavity wall. It is readily seen that the unknown dependent variables involved here are σ_{rr}, $\sigma_{\theta\theta}$ and U_r ($\sigma_{r\theta} = U_\theta = 0$) which are the functions of r only. Also from equation (30), when x and z are relaced by r and θ, it is seen that the only possible G's are $G_1^{\pm} = \pm[(\lambda+2\mu)/\rho]^{1/2}$. It follows from equations (38) through (43) that the characteristic equations for this problem are

$$\mathrm{d}\sigma_{rr}-\rho G_L^{\pm}\,\mathrm{d}U_r = \left(\sigma_{\theta\theta}-\sigma_{rr}+\rho G_L^{\pm}\frac{\lambda}{\lambda+2\mu}U_r\right)\frac{\mathrm{d}r}{r}$$

along $\mathrm{d}r/\mathrm{d}t = G_L$ where

$$G_L^{\pm} = \pm\sqrt{\left(\frac{\lambda+2\mu}{\rho}\right)}$$

$$\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}\sigma_{\theta\theta}} = \frac{\lambda+2\mu}{\lambda}\left[1-\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{U_r}{r}\frac{\mathrm{d}t}{\mathrm{d}\sigma_{\theta\theta}}\right]$$

along $dr = 0$ which are indeed the equations (18) and (19) on page 161 in Chou and Koenig's paper [12].

A two-spatial dimensional problem, for instance, is a circular cylindrical cavity subjected to a line load. Miklowitz [13] solved this problem by the transform techniques. The problem, when attempted by the theory of characteristics, seems to lead itself to the cylindrical characteristics equations.

9. CONCLUSIONS AND DISCUSSION

A method has been established by which differential equations are derived governing the propagation of discontinuities along the bicharacteristic curves. These differential equations are known as the characteristic equations. This method which makes use of Hadamard's kinematical discontinuity relations, offers an alternative approach to the conventional directional derivative approach.

The characteristic equations were derived with reference first to Cartesian and then to cylindrical coordinate systems for a linear elastic, isotropic, and homogeneous material where plane deformation is prevalent. The applicability of these equations is confined to cases where discontinuities may occur in the derivatives of the velocities and stresses, while these variables remain continuous.

The derived characteristic equations for two-spatial dimensions were shown to be applicable equally well to one-spatial dimensional wave problems. Also, the Cartesian characteristic equations, derived here, are in complete agreeement with Clifton's [2] equations derived by the conventional approach.

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REFERENCES

[1] J. D. ACHENBACH, Wave propagation in a three-parameter viscoelastic medium. Department of Aeronautics and Astronautics, Stanford University, Sudaer No. 132, July 1962.

[2] R. J. CLIFTON, The difference method for plane problem in dynamic elasticity. *Q. appl. Math.* **25**, 97–116 (1967).

[3] P. C. CHOU and R. R. KARPP, Two-dimensional dynamic elasticity problems by the numerical method of characteristics. *Proc. Army Symposium on Solid Mechanics*, September 1968. AMMRCVMS Mechanics Research Center, Watertown, Massachusetts, pp. 111–126.

[4] D. J. RICHARDSON, The solution of two-dimensional hydrodynamic equations by the method of characteristics. *Methods in Computational Physics*, edited by BERNI ADLER, Vol. 3. Academic Press (1964).

[5] JACQUES HADAMARD, *Leçons sur la Propagation des Ondes et les Équations de L'hydrodynamique*. Chelsea Publishing Company (1949).

[6] TULLIO LEVI-CIVITA, *Caractéristiques des Systèmes Différentiels et Propagation des Ondes*. Librairie Félix Alcan (1932).

[7] T. Y. THOMAS, Singular surfaces and flow lines in the theory of plasticity. *J. ration. Mech. Analysis* **2**, 339–381 (1953).

[8] R. HILL, Discontinuity relations in mechanics of solids. *Progress in Solid Mechanics*, edited by SNEDDON and HILL, Vol. 2. North-Holland (1961).

[9] A. M. FREUDENTHAL and H. GEIRINGER, The mathematical theories of the inelastic continuum. *Handbuch der Physik*, edited by S. FLÜGGE, Vol. 6. Springer (1958).

[10] P. R. GARABEDIAN, *Partial Differential Equations*. Wiley (1964).
[11] T. Y. THOMAS, The decay of waves in elastic solids. *J. math. Mech.* **6**, 759–768 (1957).
[12] P. C. CHOU and H. A. KOENIG, A unified approach to cylindrical and spherical elastic waves by method of characteristics. *J. appl. Mech.* **33**, 159–168 (1966).
[13] J. MIKLOWITZ, Pulse propagation in a viscoelastic solid with geometric dispersion, in *Stress Waves in Anelastic Solids*, pp. 255–276, edited by HERBERT KOLSKY and WILLIAM PRAGER. Springer (1964).

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Абстракт—Представляется метод уточняющий теорию характеристик для учета упругих волн в дву пространственных размерах. Он дает возможность применить работу Адамара, касающейся поверхностей разры ва в зависимых переменных и их производных.

Приводится модель поверхности распространения в линейной, упругой, изотропной и упругой среде. Скорости и напряжения материальных уастиц разрывны в своих первых частных производных, поперек поверхности распространения. Используется преобразование этой модели, благодаря которому применяются соответствующие зависимости для разрыва, вытекающие из теории Адамара, для динамических уравнений поля. Применение этих результатов преобразования в систему необходимых динамических условий приводит затем к выводу уравнений характеристик для дву пространственных измерений.